

SAMPLING ERROR ANALYSIS AND PROPERTIES OF NON-BANDLIMITED SIGNALS THAT ARE RECONSTRUCTED BY GENERALIZED SINC FUNCTIONS

YOUFA LI, QIUHUI CHEN, TAO QIAN AND YI WANG

ABSTRACT. Recently efforts have been made to use generalized sinc functions to perfectly reconstruct various kinds of non-bandlimited signals. As a consequence, perfect reconstruction sampling formulas have been established using such generalized sinc functions. One of our goals in this paper is to study the error when a truncation scheme is employed for this type of sampling formula which involves an infinite sum. This will be done in Section 3. We further analyze the error of the sampling formula when there are noises present in the samples in Section 4. Finally we discuss the reproducing properties and Sobolev smoothness of functions in the space of non-bandlimited signals that admits such a sampling formula. We begin with a discussion of properties of the generalized sinc function in Section 2.

1. INTRODUCTION

We begin by establishing some notations used in the paper. Let \mathbb{N} be the set of natural numbers, \mathbb{Z} be the set of integers, and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For a positive integer $n \in \mathbb{N}$, we use the index set $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$. Furthermore, we denote by \mathbb{R} the set of real numbers, and by \mathbb{C} the set of complex numbers. Let X be a subset of \mathbb{R} , and for $q \in \mathbb{N}$, we say a function f is in $L^q(X)$ if and only if

$$\|f\|_{q,X} := \left(\int_X |f(t)|^q dt \right)^{1/q} < \infty,$$

and f is said to be in $L^\infty(X)$ if

$$\|f\|_{\infty,X} := \text{ess sup}\{|f(t)| : t \in X\} < \infty.$$

Key words and phrases. generalized sinc function, non-bandlimited signal, sampling theorem, truncated error, noisy error, reproducing property, Sobolev smoothness.

Li is supported by the National Natural Science Foundation of China (Grant No.11126343), Natural Scientific Project of Guangxi University under grant XBZ110572 and by Macao Science and Technology Fund FDCT/056/2010/A3 for his Postdoctoral research. Chen is supported by NSFC under grant 61072126 and by Natural Science Foundation of Guangdong Province under grant S2011010004986. Qian is supported by Grant of University of Macau UL017/08-Y3/MAT/QT01/FST and by Macao Science and Technology Fund FDCT/056/2010/A3.

Similarly, let Z be a subset of \mathbb{Z} , a sequence $\mathbf{y} := (y_k : k \in Z)$ is said to be in $l^q(Z)$ if and only if

$$\|\mathbf{y}\|_{q,Z} := \left(\sum_{k \in Z} |y_k|^q \right)^{1/q} < \infty,$$

and \mathbf{y} is in $l^\infty(Z)$ if

$$\|\mathbf{y}\|_{\infty,Z} := \sup\{|y_k| : k \in Z\} < \infty.$$

In digital signal processing, the classic *sinc* function is fundamentally significant due to the Whittaker-Kotelnikov-Shannon (WKS) sampling theorem [16, 17, 2]. Recall that the classic sinc function is defined at $t \in \mathbb{R}$ by the equation

$$\text{sinc}(t) := \frac{\sin t}{t}.$$

The WKS sampling theorem enables to reconstruct a bandlimited signal from shifts of sinc functions weighted by the uniformly spaced samples of that signal. It is natural to ask whether similar sampling theorem exists for *non-bandlimited signals*. To that end, recently efforts have been made to extend the classic sinc to *generalized sinc functions*, for example, in [4, 5, 6]. One kind of generalized sinc functions given in [4], denoted by sinc_F , is defined as the *inverse Fourier transform* of a so-called *symmetric cascade filter*, denoted by H . Specifically,

$$(1.1) \quad \text{sinc}_F := \sqrt{\frac{\pi}{2}} \mathcal{F}^{-1} H,$$

where for any signal $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} dt.$$

The symmetric cascade filter H is a piecewise constant function whose value at $\xi \in \mathbb{R}$ is given by

$$(1.2) \quad H(\xi) := \sum_{n \in \mathbb{Z}_+} b_n \chi_{I_n}(\xi),$$

where the sequence $\mathbf{b} = (b_n : n \in \mathbb{Z}_+)$ is in $l^2(\mathbb{Z}_+)$, χ_I is the *indicator function* of the set I , and the interval I_n , $n \in \mathbb{Z}_+$, is the union of two symmetric intervals given by the equation

$$I_n := (-(n+1), -n] \cup [n, (n+1)).$$

Of course, we have that $H \in L^2(\mathbb{R})$ because $\mathbf{b} \in l^2(\mathbb{Z}_+)$, and hence $\text{sinc}_F \in L^2(\mathbb{R})$ since the Fourier operator is closed in $L^2(\mathbb{R})$.

With the generalized function sinc_F , a perfect reconstruction sampling theorem was established in [4] for the purpose of reconstructing non-bandlimited signals. This kind of reconstruction sampling theorem may be very useful to study signals with polynomial decaying Fourier spectra that arise in evolution equations and control theories [13, 14].

One of our goals in this paper is to study the error when a truncation scheme is employed for the above sampling formula which involves an infinite sum. This will be done in Section 3. We further analyze the error of the sampling formula when there are noises present in the

samples in Section 4. Finally we discuss the reproducing properties and Sobolev smoothness of functions in the space of non-bandlimited signals that admits such a sampling formula. We begin with a discussion of properties of the function sinc_F in Section 2.

2. PROPERTIES OF THE GENERALIZED SINC FUNCTIONS

Surprisingly the function sinc_F has many properties that are similar to the classic sinc, such as cardinal, orthogonal properties and it behaves also similarly to the classic sinc. In the special case, the function sinc_F reduces to the classic sinc. Let us first review the approach to obtain an explicit form of sinc_F .

The symmetric cascade filter H can be associated with an analytic function F on the open unit disk

$$\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}.$$

The value of F at $z \in \Delta$ is well defined by

$$(2.1) \quad F(z) := \sum_{n \in \mathbb{Z}_+} b_n z^n,$$

as $\mathbf{b} \in l^2(\mathbb{Z}_+)$. Recall that the Hardy space $H^2(\Delta)$ consists of all functions f analytic in Δ , with norm given by

$$\|f\|_{H^2(\Delta)}^2 = \sup_{r \in (0,1)} \frac{1}{2\pi} \int_{[-\pi, \pi]} |f(re^{it})|^2 dt.$$

Since, by hypothesis, $\mathbf{b} \in l^2(\mathbb{Z}_+)$, we have that $F \in H^2(\Delta)$. Consequently its extension to the boundary $\partial\Delta$ of Δ is in $L^2(\partial\Delta)$.

Thus, from equations (1.1), (1.2) and (2.1) an explicit form of $\text{sinc}_F(t)$, $t \in \mathbb{R}$ can be found as

$$(2.2) \quad \text{sinc}_F(t) = \text{sinc}\left(\frac{t}{2}\right) \text{Re} \left\{ F(e^{it}) e^{\frac{1}{2}it} \right\}, \quad \text{a.e.}$$

where $\text{Re}(z)$ is the real part of a complex number z .

We observe that if $\mathbf{b} \in l^1(\mathbb{Z}_+)$ then $H \in L^1(\mathbb{R})$ and F is continuous on the boundary of Δ , which in turn implies sinc_F is continuous and bounded.

A very interesting fact, as discovered in the paper [3], is that when F is imposed with a *stronger condition* of having analyticity in a neighborhood of the closed unit disc $\overline{\Delta}$, the function sinc_F can be generated through a function, denoted by G , that is also analytic in a neighborhood of the closed unit disc, real on the real axis and normalized so that $G(1) = 1$ and $G'(1) \neq 0$. The function G is linked to F by the equation

$$(2.3) \quad F(z) := \frac{G(z) - 1}{z - 1}.$$

A real-valued function ϕ_G whose value at $t \in \mathbb{R}$ is then defined through the imaginary part of the values of G on the unit circle by the equation

$$(2.4) \quad \phi_G(t) := \frac{\text{Im}(G(e^{it}))}{t}.$$

Applying equation (2.2) to compute sinc_F by using equations (2.3) and (2.4) yields for all $t \in \mathbb{R}$,

$$(2.5) \quad \text{sinc}_F(t) = \phi_G(t).$$

Two important examples can be demonstrated for this construction. When $G = z$, i.e., $F = 1$, we have $\text{sinc}_F = \phi_G = \text{sinc}$. For the second example, let G be the *Blaschke product* of order $n \in \mathbb{N}$ with parameters $\mathbf{a} := (a_j : j \in \mathbb{Z}_n) \in (-1, 1)^n$, that is,

$$G(z) = B_{\mathbf{a}}(z) := \prod_{j \in \mathbb{Z}_n} \frac{z - a_j}{1 - \overline{a_j}z}.$$

Then $\text{sinc}_F(t) = \phi_G(t) = \frac{\sin \theta_{\mathbf{a}}(t)}{t}$, where $\theta_{\mathbf{a}}$ is determined by the boundary value of the Blaschke product at $t \in \mathbb{R}$ by $e^{i\theta_{\mathbf{a}}(t)} = B_{\mathbf{a}}(e^{it})$.

We next list some properties of the function sinc_F .

Proposition 2.1. *Let the generalized function sinc_F be defined by equation (2.2). Then*

- (1) $\text{sinc}_F(n\pi) = F(1)\delta_{n,0}$, where $\delta_{n,0} = 1$ if $n = 0$ and $\delta_{n,0} = 0$ if $n \in \mathbb{Z} \setminus \{0\}$.
- (2) sinc_F is bounded, infinitely differentiable.
- (3) $|\text{sinc}_F(t)| \leq \frac{4\|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}}{2+|t|}$, for $t \in \mathbb{R}$, and $\text{sinc}_F \in L^2(\mathbb{R})$.
- (4) The set $\{\text{sinc}_F(\cdot - n\pi) : n \in \mathbb{Z}\}$ is an orthogonal set, that is

$$\langle \text{sinc}_F, \text{sinc}_F(\cdot - n\pi) \rangle = \pi \|\mathbf{b}\|_{l^2(\mathbb{Z}_+)}^2 \delta_{n,0},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on the Hilbert space $L^2(\mathbb{R})$.

Proof. The first two statements directly follow from equation (2.2). The third statement follows from equation (2.2) and noticing $\text{sinc}(t) \leq \frac{2}{1+|t|}$ for any $t \in \mathbb{R}$. The fourth statement is a special case of Corollary 3.2 of [4]. For the convenience of readers, we provide a direct proof here. By Parseval's theorem and equation (1.1) we have

$$\begin{aligned} \int_{\mathbb{R}} \text{sinc}_F(t) \text{sinc}_F(t - n\pi) dt &= \frac{\pi}{2} \int_{\mathbb{R}} H^2(x) e^{in\pi x} dx = \frac{\pi}{2} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}_+} b_k^2 \chi_{I_k}(x) e^{in\pi x} dx \\ &= \frac{\pi}{2} \sum_{k \in \mathbb{Z}_+} b_k^2 \int_{I_k} e^{in\pi x} dx = \pi \left(\sum_{k \in \mathbb{Z}_+} b_k^2 \right) \delta_{n,0}, \end{aligned}$$

where, in the last equality we have used the orthogonality of the set $\{e^{-in\pi x} : n \in \mathbb{Z}\}$ on I_k , $k \in \mathbb{Z}_+$. The interchange of the integral operator and the infinite sum is guaranteed by the convergence of the series. □

3. SAMPLING TRUNCATION ERROR ANALYSIS

In [4], a Shannon-type sampling theorem is given concerning functions in the shift-invariant space

$$V_F := \left\{ \sum_{n \in \mathbb{Z}} c_n \text{sinc}_F(\cdot - n\pi) : F(1) = 1, \mathbf{c} = (c_n : n \in \mathbb{Z}) \in l^2(\mathbb{Z}) \right\}$$

of sinc_F . The Shannon-type sampling theorem is the direct result of the properties in the previous proposition. We record it here.

Theorem 3.1. *A signal $f \in V_F$ if and only if*

$$(3.1) \quad f = \sum_{n \in \mathbb{Z}} f(n\pi) \text{sinc}_F(\cdot - n\pi).$$

Equation (3.1) necessarily implies that the sampling sequence $(f(n\pi) : n \in \mathbb{Z}) \in l^2(\mathbb{Z})$ by the orthogonality of the set $\{\text{sinc}_F(\cdot - n\pi) : n \in \mathbb{Z}\}$. The above equation of course is true in $L^2(\mathbb{R})$ norm. However, if $\mathbf{b} \in l^1(\mathbb{Z}_+)$, equation (3.1) holds true pointwise, because by Cauchy-Schwartz inequality the series on the right side of equation (3.1) converges uniformly, hence the limiting function f is continuous.

We remark that, as pointed out in [4], a function $f \in V_F$ can be characterized by its spectrum. Specifically, a function $f \in V_F$ if and only if

$$(3.2) \quad \mathcal{F}f(\xi) = \sqrt{\frac{\pi}{2}} \left(\sum_{n \in \mathbb{Z}} f(n\pi) e^{-in\pi\xi} \right) H(\xi).$$

Equation (3.2) holds true in $L^2(\mathbb{R})$ if $\mathbf{b} \in l^2(\mathbb{R})$, and a.e. pointwise if $f \in L^1(\mathbb{R})$ and the sample sequence $(f(n\pi) : n \in \mathbb{Z}) \in l^1(\mathbb{Z})$.

The following property is true for functions in the space V_F that is similar to Parseval's identity.

Proposition 3.2. *If $f \in V_F$ then*

$$(3.3) \quad \|f\|_{L^2(\mathbb{R})}^2 = \pi \|\mathbf{b}\|_{l^2(\mathbb{Z}_+)}^2 \sum_{n \in \mathbb{Z}} f^2(n\pi)$$

Proof. This is a direct result of equation (3.1) and Proposition 2.1 (4). \square

The recovering formula in (3.1) involves an infinite sum. In practice, we need to truncate the series to approximate f . Here, we prefer an *adaptively truncated sum* and offer a pointwise estimation of the error.

For fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}$, define the index set

$$J_n(t) := \{j : j \in \mathbb{Z}, |t - j\pi| \leq n\pi\}$$

and the partial sum given at $t \in \mathbb{R}$ by

$$(3.4) \quad S_n(t) := \sum_{j \in J_n(t)} f(j\pi) \text{sinc}_F(t - j\pi).$$

The adaptive truncation strategy allows that for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$, there are approximately $2n$ functions sinc_F shifted by a distance of $j\pi$, $j \in J_n(t)$, on both sides of t . The following theorem states that the truncated error estimate is $\mathcal{O}(n^{-1/2})$. We recall a Calculus fact that will be used a couple of times later. For a positive and decreasing sequence $\mathbf{a} := (a_n : n \in \mathbb{Z}_+)$, if $\frac{1}{a_{n+1}} - \frac{1}{a_n} = c$, where c is a positive constant, then

$$(3.5) \quad \sum_{n \in \mathbb{Z}_+} \frac{1}{a_n^2} \leq \frac{1}{a_0^2} + \frac{1}{c} \int_{[a_0, \infty)} \frac{1}{x^2} dx.$$

Theorem 3.3. *Let $f \in V_F$ and $\mathbf{b} \in l^1(\mathbb{Z}_+)$. Then we have*

$$(3.6) \quad \sup_{t \in \mathbb{R}} |f(t) - S_n(t)| \leq \frac{\|\mathbf{b}\|_{\ell^1(\mathbb{Z}_+)}}{\|\mathbf{b}\|_{\ell^2(\mathbb{Z}_+)}} \|f\|_{L^2(\mathbb{R})} \sqrt{\frac{8}{\pi^3} \left(\frac{1}{n^2} + \frac{1}{n} \right)}.$$

Proof. We first note that by equation (2.2), for $t \in \mathbb{R}$,

$$(3.7) \quad |\text{sinc}_F(t)| \leq \|\mathbf{b}\|_{l^1(\mathbb{Z}_+)} \left| \text{sinc} \frac{t}{2} \right| \leq \begin{cases} \|\mathbf{b}\|_{l^1(\mathbb{Z}_+)} & \text{if } |t| \leq 2, \\ \frac{\|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}}{|t/2|} & \text{if } |t| \geq 2. \end{cases}$$

By Cauchy-Schwartz inequality and equation (3.7), we have

$$(3.8) \quad \begin{aligned} |f(t) - S_n(t)|^2 &= \left| \sum_{j \in \mathbb{Z} \setminus J_n(t)} f(j\pi) \text{sinc}_F(t - j\pi) \right|^2 \\ &\leq \left(\sum_{j \in \mathbb{Z} \setminus J_n(t)} f^2(j\pi) \right) \left(\sum_{j \in \mathbb{Z} \setminus J_n(t)} \text{sinc}_F^2(t - j\pi) \right) \\ &\leq 4 \|\mathbf{b}\|_{\ell^1(\mathbb{Z}_+)}^2 \left(\sum_{j \in \mathbb{Z} \setminus J_n(t)} f^2(j\pi) \right) \left(\sum_{j \in \mathbb{Z} \setminus J_n(t)} \frac{1}{(t - j\pi)^2} \right). \end{aligned}$$

We next estimate for $t \in \mathbb{R}$ the value of

$$(3.9) \quad R_n(t) := \sum_{j \in \mathbb{Z} \setminus J_n(t)} \frac{1}{(t - j\pi)^2}.$$

It is easy to see that R_n is periodic with period π . For $t \in [0, \pi)$, using equation (3.5) we have

$$(3.10) \quad R_n(t) \leq 2 \left(\frac{1}{n^2 \pi^2} + \frac{1}{\pi} \int_{[n\pi, \infty)} \frac{1}{x^2} dx \right) = \frac{2}{\pi^2} \left(\frac{1}{n^2} + \frac{1}{n} \right).$$

Using equation (3.3), we obtain that the quantity

$$(3.11) \quad \sum_{j \in \mathbb{Z} \setminus J_n(t)} f^2(j\pi) \leq \sum_{n \in \mathbb{Z}} f^2(n\pi) = \frac{\|f\|_{L^2(\mathbb{R})}^2}{\pi \|\mathbf{b}\|_{\ell^2(\mathbb{Z}_+)}^2}.$$

Finally we conclude (3.6) by combining equations (3.8), (3.10) and (3.11). \square

4. ERROR ANALYSIS WHEN NOISES PRESENT

Theorem 3.1 establishes that a signal in the space V_F can be perfectly reconstructed by an infinite sum of shifts of the generalized sinc functions weighted by equally spaced samples of that signal. However, samples are often corrupted by noise in practice. In [15], Smale and Zhou gave an error estimate in the probability sense for Shannon sampling theorem with noised samples. In [1], Aldroubi, Leonetti and Sun studied the error by frame reconstruction from noised samples. In this section, we shall investigate the error of the

sampling formula (3.1) with noised samples. Specifically, we deal with the following noise model concerning the noisy samples $\tilde{f}(n\pi)$, $n \in \mathbb{Z}$ whose value corrupted by noise is given by

$$(4.1) \quad \tilde{f}(n\pi) = f(n\pi) + \epsilon(n\pi), \quad n \in \mathbb{Z},$$

where we assume that $(\epsilon(n\pi) : n \in \mathbb{Z})$ is a sequence of *independent and identically distributed* random variables with the expectation and variance of each given by

$$(4.2) \quad \mathbb{E}(\epsilon(n\pi)) = 0, \quad \text{Var}(\epsilon(n\pi)) = \sigma^2, \quad n \in \mathbb{Z}.$$

Thus in practice, we recover $f \in V_F$ by

$$(4.3) \quad f^\natural = \sum_{n \in \mathbb{Z}} \tilde{f}(n\pi) \text{sinc}_F(\cdot - n\pi).$$

We next study the expectation $\mathbb{E}(f - f^\natural)$ and variance $\text{Var}(f - f^\natural)$.

Theorem 4.1. *Let $f \in V_F$ be recovered by equation (4.3) with the noised samples $\tilde{f}(n\pi)$ being referred to in (4.1). Then*

$$\mathbb{E}(f(t) - f^\natural(t)) = 0$$

and

$$(4.4) \quad \text{Var}(f(t) - f^\natural(t)) \leq 2\sigma^2 \|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}^2 \left(1 + \frac{8}{\pi^2}\right).$$

Proof. We first compute the expectation $\mathbb{E}(f(t) - f^\natural(t))$.

$$\begin{aligned} \mathbb{E}(f(t) - f^\natural(t)) &= \mathbb{E} \left(\sum_{n \in \mathbb{Z}} \epsilon(n\pi) \text{sinc}_F(t - n\pi) \right) \\ &= \sum_{n \in \mathbb{Z}} \mathbb{E}(\epsilon(n\pi)) \text{sinc}_F(t - n\pi) \\ &= 0. \end{aligned}$$

Invoking the assumed independence of $\epsilon(j\pi)$, $j \in \mathbb{Z}$ we obtain

$$\begin{aligned} \text{Var}(f(t) - f^\natural(t)) &= \text{Var} \left(\sum_{n \in \mathbb{Z}} \epsilon(n\pi) \text{sinc}_F(t - n\pi) \right) \\ &= \sum_{n \in \mathbb{Z}} \text{Var}(\epsilon(n\pi) \text{sinc}_F(t - n\pi)) \\ &= \sigma^2 \sum_{n \in \mathbb{Z}} \text{sinc}_F^2(t - n\pi). \end{aligned} \tag{4.5}$$

The sequence $\sum_{n \in \mathbb{Z}} \text{sinc}_F^2(t - n\pi)$ can be easily estimated. Note it is periodic with period π .

Recalling equation (3.7), for $t \in [0, \pi)$, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \text{sinc}_F^2(t - n\pi) &= \text{sinc}_F^2(t) + \text{sinc}_F^2(t - \pi) + \sum_{n \in -\mathbb{N}} \text{sinc}_F^2(t - n\pi) + \sum_{n \in 1+\mathbb{N}} \text{sinc}_F^2(t - n\pi) \\ &\leq 2 + \sum_{n \in -\mathbb{N}} \text{sinc}_F^2(t - n\pi) + \sum_{n \in 1+\mathbb{N}} \text{sinc}_F^2(t - n\pi). \end{aligned}$$

Noting for $t \in [0, \pi)$, $n \in \mathbb{N}$, $\frac{|t - n\pi|}{2} \geq \frac{\pi}{2}$ and in view of equation (3.5) we obtain that

$$\sum_{n \in 1+\mathbb{N}} \text{sinc}_F^2(t - n\pi) \leq \|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}^2 \left(\frac{4}{\pi^2} + \frac{2}{\pi} \int_{[\frac{\pi}{2}, \infty)} \frac{1}{x^2} dx \right) = \|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}^2 \left(\frac{8}{\pi^2} \right).$$

Similarly we have

$$\sum_{n \in -\mathbb{N}} \text{sinc}_F^2(t - n\pi) \leq \|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}^2 \left(\frac{8}{\pi^2} \right).$$

Consequently we obtain that

$$(4.6) \quad \sum_{n \in \mathbb{Z}} \text{sinc}_F^2(t - n\pi) \leq 2\|\mathbf{b}\|_{l^1(\mathbb{Z}_+)}^2 \left(1 + \frac{8}{\pi^2} \right).$$

Finally combining equations (4.5) and (4.6) proves equation (4.4). \square

5. THE REPRODUCING PROPERTY AND SOBOLEV SMOOTHNESS

When the analytical function F is chosen to be $F = 1$, the space V_F reduces to the space of bandlimited signals. The space of bandlimited signals is a reproducing kernel Hilbert space (r.k.H.s) [7]. We next show that the space V_F has a similar property. However, the reproducing kernel is a distribution in the space of tempered distributions. We define the distribution for $x, t \in \mathbb{R}$ by

$$\Phi(x, t) := \frac{1}{\pi} \text{sinc}_F(t - x) \sum_{k \in \mathbb{Z}} e^{i2kx}.$$

Recalling the Poisson formula in the distribution form:

$$\sum_{k \in \mathbb{Z}} e^{-i2kx} = \pi \sum_{k \in \mathbb{Z}} \delta(x - k\pi),$$

where δ is the usual Dirac delta function, we immediately obtain an alternative form of Φ given by

$$(5.1) \quad \Phi(x, t) = \sum_{k \in \mathbb{Z}} \delta(x - k\pi) \text{sinc}_F(t - x).$$

Theorem 5.1. *Let $f \in V_F$ then*

$$(5.2) \quad f(t) = \int_{\mathbb{R}} f(x) \Phi(x, t) dx$$

in the distribution sense.

Proof. Poisson's summation formula indicates that

$$(5.3) \quad \sum_{n \in \mathbb{Z}} f(n\pi) e^{-in\pi\xi} = \sqrt{\frac{2}{\pi}} \sum_{k \in \mathbb{Z}} \mathcal{F}f(\xi + 2k).$$

Thus equation (3.2) is equivalent to

$$(5.4) \quad \mathcal{F}f(\xi) = \sum_{k \in \mathbb{Z}} \mathcal{F}f(\xi + 2k) H(\xi)$$

This leads to

$$\begin{aligned} f(t) &= \mathcal{F}^{-1} \left(\sum_{k \in \mathbb{Z}} \mathcal{F}f(\cdot + 2k) H \right) (t) \\ &= \sum_{k \in \mathbb{Z}} \mathcal{F}^{-1} (\mathcal{F}f(\cdot + 2k) H) (t) \end{aligned}$$

The interchange of the order of the sum and integral operator is justified by the convergence of the series. Recalling equations (1.1) and the convolution theorem for Fourier transform, we therefore conclude that

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} \left((\mathcal{F}^{-1}(\mathcal{F}f(\cdot + 2k))) * (\mathcal{F}^{-1}H) \right) (t) \\ &= \sum_{k \in \mathbb{Z}} \frac{1}{\pi} \int_{\mathbb{R}} f(x) e^{-i2kx} \text{sinc}_F(t - x) dx \\ &= \frac{1}{\pi} \int_{\mathbb{R}} f(x) \sum_{k \in \mathbb{Z}} e^{-i2kx} \text{sinc}_F(t - x) dx \end{aligned}$$

which is equation (5.2). □

Next we discuss the *Sobolev smoothness* of a function in the space V_F . We say that a function f belongs to the Sobolev space $H^s(\mathbb{R})$ if

$$(5.5) \quad \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty.$$

The *Sobolev smoothness* of f is defined to be $\nu_2(f) := \sup\{s : f \in H^s(\mathbb{R})\}$. As an example, the Sobolev smoothness of a function in the space of bandlimited signals is infinity. Algorithms [9, 8, 10, 12] or even Matlab routines [11] are given for calculating Sobolev smoothness of a refinable function. However, those algorithms are not applicable to calculate the Sobolev smoothness of a function in the space V_F . We give a characterization in the next theorem about the Sobolev smoothness of the functions in the space V_F .

Theorem 5.2. *If $f \in V_F$ and for some $s \in \mathbb{R}$,*

$$(5.6) \quad \sum_{n \in \mathbb{Z}_+} b_n^2 n^{2s} < \infty$$

then the Sobolev smoothness of the function f satisfies $\nu_2(f) \geq s$.

Proof. By equations (1.2) and (3.2) we obtain that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi = \sum_{n \in \mathbb{Z}_+} b_n^2 \int_{I_n} \left(\sum_{k \in \mathbb{Z}} f(k\pi) e^{-ik\pi\xi} \right)^2 (1 + \xi^2)^s d\xi.$$

Noting when $\xi \in I_n$, $n \leq |\xi| < n+1$ for each $n \in \mathbb{Z}_+$, which implies that $1 + \xi^2 \leq 3n^2$, for $n \in \mathbb{Z}_+$. Consequently, we deduce that

$$\begin{aligned} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi &\leq \sum_{n \in \mathbb{Z}_+} 3^s b_n^2 n^{2s} \int_{I_n} \left(\sum_{k \in \mathbb{Z}} f(k\pi) e^{-ik\pi\xi} \right)^2 d\xi \\ &= \sum_{n \in \mathbb{Z}_+} 3^s b_n^2 n^{2s} \sum_{k \in \mathbb{Z}} f^2(k\pi), \end{aligned}$$

where in the last equality again we have used the orthogonality of the set $\{e^{-ik\pi\xi} : k \in \mathbb{Z}\}$ on I_n , $n \in \mathbb{Z}_+$. Now apply equation (3.3) we obtain that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \leq \frac{3^s \|f\|_{L^2(\mathbb{R})}^2}{\pi \|\mathbf{b}\|_{l^2(\mathbb{Z}_+)}^2} \sum_{n \in \mathbb{Z}_+} b_n^2 n^{2s} < \infty$$

by the assumption (5.6). \square

For example, consider the space V_F associated with the analytic function F in equation (2.1) such that the sequence \mathbf{b} is a geometric sequence with $b_n = a^n(1-a)$, $n \in \mathbb{Z}_+$, where $a \in (-1, 1)$. The series $\sum_{n=0}^{\infty} n^{2s} a^{2n} (1-a)^2$ converges for every $s \in \mathbb{R}$. Therefore, we have the Sobolev smoothness $\nu_2(f) = \infty$. In particular, when $a = 0$, the space V_F degenerates to the space of bandlimited functions, and its Sobolev smoothness is infinity.

If \mathbf{b} is given by $b_n = \frac{1}{(n+1)^3}$, $n \in \mathbb{Z}_+$, the series $\sum_{n=0}^{\infty} n^{2s} \frac{1}{(2n+1)^6}$ converges if and only if $6 - 2s > 1$, that is, $s < 2.5$, which implies that $\nu_2(f) = 2.5$ in this case.

REFERENCES

- [1] A. Aldroubi, C. Leonetti, and Q. Sun. Error analysis of frame reconstruction from noisy samples. *IEEE Transactions on Signal Processing*, 56:2311–2325, 2008.
- [2] P.L. Butzer, W. Engels, and S. Ries. The Shannon sampling series and the reconstruction of signals in terms of linear, quadratic and cubic splines. *SIAM J. Appl. Math.*, 46:299–323, 1986.
- [3] Q. Chen and C.A. Micchelli. The Bedrosian identity for functions analytic in a neighborhood of the unit circle. *Complex Anal. Oper. Theory*, Online First, 2011.
- [4] Q. Chen, C.A. Micchelli, and Y. Wang. Functions with spline spectra and their applications. *Int. J. Wavelets Multiresolut. Inf. Process.*, 8(2), 2010.
- [5] Q. Chen and T. Qian. Sampling theorem and multi-scale spectrum based on non-linear Fourier atoms. *Applicable Analysis*, 88(6):903–919, 2009.
- [6] Q. Chen, Y.B. Wang, and Y. Wang. A sampling theorem for non-bandlimited signals using generalized sinc functions. *Comput. Math. Appl.*, 56:1650–1661, 2008.
- [7] I. Daubechies. *Ten Lectures on Wavelets*, volume 61 of *CBMS*. SIAM, Philadelphia, 1992.
- [8] B. Han. Computing the smoothness exponent of a symmetric multivariate refinable function. *SIAM J. Matrix Anal. Appl.*, 24:693–714, 2003.

- [9] B. Han. Vector cascade algorithms and refinable function vectors in Sobolev spaces. *J. Approx. Theory*, 24:693–714, 2003.
- [10] B. Han. Construction of wavelets and framelets by the projection method. *Int. J. Appl. Math. Appl.*, 1:1–40, 2008.
- [11] Q.T. Jiang. Matlab routines for Sobolev and Hölder smoothness computation of refinable functions. Technical report, 2001.
- [12] Y. Li. Hermite-like interpolating refinable function vector and its application in signal recovering. *J. Fourier Analysis Appl.*, Online First, 2011. doi: 10.1007/s00041-011-9208-z.
- [13] Z. Liu and B. Rao. Characterization of polynomial decay rate for the solution of linear evolution equation. *Zeitschrift Für Angewandte Mathematik und Physik*, 56:630–644, 2005.
- [14] K.D. Phung. Polynomial decay rate for the dissipative wave equation. *J. Differentiable Equations*, 240:92–124, 2007.
- [15] D.X. Zhou S. Smale. Shannon sampling and function reconstruction from point values. *Bulletin-American Mathematical Society*, 41:279–305, 2004.
- [16] C. Shannon. A mathematical theory of communication. *Bell Sys. Tech. J.*, 27:379–423, 1948.
- [17] C. Shannon. Communication in the presence of noise. *Proc. IRE*, 37:10–21, 1949.

YOUFA LI, COLLEGE OF MATHEMATICS AND INFORMATION SCIENCES, GUANGXI UNIVERSITY, NAN-NING, CHINA; DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF MACAU, TAIPA, MACAO, CHINA.

E-mail address: youfalee@hotmail.com

QIUHUI CHEN, CISCO SCHOOL OF INFORMATICS, GUANGDONG UNIVERSITY OF FOREIGN STUDIES, GUANGZHOU, CHINA.

E-mail address: chenqiuahui@hotmail.com

TAO QIAN, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF MACAU, TAIPA, MACAO, CHINA.

E-mail address: fsttq@umac.mo

YI WANG, DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY AT MONTGOMERY, P.O. BOX 244023, MONTGOMERY, AL 36124-4023 USA.

E-mail address: ywang2@aum.edu